The Power Model of Fitts’ Law Does not Encompass the Logarithmic Model

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Plan

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Fitts’ Law

A model of human pointing movement in HCI

One-dimensional single-shot movement paradigm

- $T$ time required to rapidly move to a target interval
- $D$ distance to the target
- $W$ size of the target

Mathematical formulation

- $T$ is linearly dependent on an index of difficulty $ID$:
  \[
  T = a + b \cdot ID
  \]

- $ID$ is a strictly increasing function of $\frac{D}{W}$:
  \[
  ID = \begin{cases} 
  \log_2 \frac{2D}{W} & \text{Fitts (1954)} \\
  \log_2 \frac{D}{W} & \text{Crossman (1956)} \\
  \log_2 \left(1 + \frac{D}{W}\right) & \text{McKenzie (1992)} \\
  \sqrt[\frac{1}{n}]{\frac{D}{W}} & \text{Meyer et al. (1988)} \\
  \left(\frac{D}{W}\right)^{1/n} & \text{Meyer et al. (1990)}
  \end{cases}
  \]
Practical range of relative distances is narrow (Guiard et al., in press):

\[
3 \lesssim \frac{D}{W} \lesssim 33 \quad \text{(accuracy saturation)}
\]

(ID = \begin{cases} 
\log_2 \frac{D}{W} \\
\log_2 (1 + \frac{D}{W}) \\
\sqrt{\frac{D}{W}} (\frac{D}{W})^{1/n}
\end{cases})

“A failure to agree for 50 years is public advertisement of a failure to disprove” Platt (1964, Strong inference, p. 351)

Mathematical Derivation of Fitts’ Law

Assuming a sequence of \( n \) submovements toward the target

**Deterministic Iterative-Corrections Model**

Crossman & Goodeve (1963)

- each submovement requires a constant time \( \Delta T \)
- and moves a constant proportion \( \lambda < 1 \) of the remaining distance

\[
T = n \Delta T \quad \text{s.t.} \quad \lambda^n D = W/2
\]

so \( T = a + b \log_2 \frac{D}{W} \).
Mathematical Derivation of Fitts’ Law

Stochastic Optimized-Submovement Model

Meyer et al. (1988,1990)

- random submovement endpoint $\Delta$ (Gaussian r.v.)
- $n = 2$ (Woodworth, 1899)
- initial ballistic submovement $T_i = \frac{D}{W s}$
  where dispersion parameter $s \propto \sigma_\Delta$
- if $|\Delta| > W/2$, secondary submovement, averaged over $\Delta$
  \[ T = \min_s \{ T_i + E|\Delta|>W/2\left(\frac{|\Delta|}{W}\right) \} \]

- Result (approx.) :
  \[ T \propto \frac{2\theta \sqrt{D/W} - \sqrt{W/D}}{\theta \sqrt{\theta} - W/D} \quad \text{where} \quad \theta \propto \exp\left(\frac{1}{2(\theta D/W - 1)}\right) \]

Meyer et al.’s Claim

for $n = 2$ :
  \[ T = a + b\sqrt{\frac{D}{W}} \]

for any $n$ :
  \[ T = a_n + b_n n^\frac{D}{W} \]

let $n \to +\infty$ :
  \[ T = a' + b' \ln\left(\frac{D}{W}\right) \]

Conclusion: The power law encompasses the logarithmic law.
Meyer et al.’s Claim is False

Proof
As \( n \to +\infty \) (Guiard et al., 2001)

\[
\sqrt[n]{\frac{D}{W}} = \exp \frac{\ln \frac{D}{W}}{n} \to 1
\]

So if there exists sequences \((a_n, b_n)\) s.t.

\[
T = a_n + b_n \sqrt[n]{\frac{D}{W}} \to a' + b' \ln \left( \frac{D}{W} \right)
\]

then \( b' = 0 \) (contradiction) Q.E.D.

Question
Validity of the power law \( T = a_n + b_n \sqrt[n]{D/W} \)?

Detailed Calculation

Let \( T = f_n(D/W) \) after \( n \) submovements. Then by the stochastic optimized-submovement model, for any \( n > 1 \):

\[
f_n(D/W) = \min_s \left\{ \frac{D/W^{-1/2}}{T_i} + E(\Delta > W/2) f_{n-1}(\frac{\Delta}{W}) \right\}
\]

To simplify, let \( \Delta \) be uniformly distributed in \((-\frac{Wz}{2}, \frac{Wz}{2})\) (Smith, 1988)

\[
f_n(y) = \min_s \left\{ \frac{y^{-1/2}}{s} + \frac{2}{s} \int_{1/2}^{s/2} f_{n-1}(x)dx \right\}
\]

By induction \( f_n \) vanishes at \( y = 1/2 \) and is strictly increasing and regular for \( y > 1/2 \).
Making the first derivative vanish, when \( s = s(y) \) is optimal:

\[
0 = -\frac{y - 1/2}{s^2} - \frac{2}{s^2} \int_{1/2}^{s/2} f_{n-1}(x) + \frac{1}{s} f_{n-1}(s) \iff f_n(y) = f_{n-1}(\frac{s}{2})
\]

so letting \( y = g_n(t) \) denote the inverse function of \( t = f_n(y) \):

\[
f_n(y) = \frac{y - 1/2}{s} + \frac{2}{s} \int_{1/2}^{s/2} f_{n-1}(x) \, dx
\]

\[
t = g_n(t) - \frac{1/2}{s} + \frac{2}{s} \int_{g_n(0)}^{g_n(t)} f_{n-1}(x) \, dx
\]

\[
t = g_n(t) - \frac{1/2}{s} + \frac{2}{s} \left( \frac{s}{2} \cdot t - \int_0^t g_{n-1}(u) \, du \right)
\]

by the inverse function integration theorem.

One obtains the relative distance \( y = D = g_n(t) \) as a function of time \( t \):

\[
g_n(t) = \frac{1}{2} + 2 \int_0^t g_{n-1}(u) \, du
\]

\[
= \frac{1}{2} + \frac{1}{2} (2t) + \frac{1}{2} \frac{(2t)^2}{2} + \cdots + \frac{1}{2} \frac{(2t)^n}{n!} \quad \text{by induction}
\]

Therefore, \( T \) is the root of the \( n \)th order equation \( \frac{D}{W} = \frac{1}{2} e_n(2T) \)

where \( e_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \) is the \( n \)th order truncated exponential. That is,

\[
T = \frac{1}{2} l_n(2 \frac{D}{W})
\]

where \( l_n = e_n^{-1} \) is the inverse function of \( e_n \).
Examples (closed form expressions)

\[ n = 2 \text{ quasi square root law (Meyer et al., 1988)} \]

\[ l_2(x) = \sqrt{2x - 1} - 1 \quad T = \sqrt{\frac{D}{W} - \frac{1}{4} - \frac{1}{2}} \]

\[ n = 3 \text{ quasi cube root law} \]

\[ l_3(x) = \frac{3\sqrt{3x + \sqrt{9x^2 - 6x + 2}} - 1}{3\sqrt{3x + \sqrt{9x^2 - 6x + 2}} - 1} \]

\[ T = \frac{1}{2} \sqrt{\frac{D}{W} + \sqrt{36\left(\frac{D}{W}\right)^2 - 12\frac{D}{W} + 2}} - \frac{1}{2} \]

\[ n = 4 \]

\[ l_4(x) = \frac{16(2x - 1)}{\sqrt{192x + 32\sqrt{32x^3 - 12x^2 + 12x - 3} - 32}} - \frac{1}{\sqrt{192x + 32\sqrt{32x^3 - 12x^2 + 12x - 3} - 32}} - 8 \]

\[ + \frac{32 - \frac{1}{\sqrt{192x + 32\sqrt{32x^3 - 12x^2 + 12x - 3} - 32}} - 4}{\sqrt{192x + 32\sqrt{32x^3 - 12x^2 + 12x - 3} - 32}} - 8 \]

A Power Law?

Meyer et al. would argue that when \( \frac{D}{W} \) is large (when \( T \) is large),

\[ \frac{D}{W} = \frac{1}{2} e^{n(2T)} \approx \frac{1}{2} \frac{(2T)^n}{n!} \]

so that we obtain a \( nth \) root (power) law:

\[ T = \frac{1}{2} \sqrt{n! \frac{D}{W}} \]

However, this is not a genuine power model since as \( n \to +\infty \),

\[ \sqrt[n]{n!} \sim \frac{n}{e} \to +\infty! \]
A Power Law?

![Power laws graph](image)

A Quasi Exponential Law

Rather, as \( n \to +\infty \),

\[
\frac{D}{W} = \frac{1}{2} e_n(2T) \to \frac{1}{2} \exp(2T)
\]

and we end up with a logarithmic law:

\[
T = \frac{1}{2} \ln \left(2 \frac{D}{W}\right)
\]
A Quasi Exponential Law

Open Issues

- robustness to endpoint $\Delta$’s distribution (uniform vs. Gaussian)
- sensitivity to the number $n$ of submovements
- a submovement theory that is well adapted to information theoretic principles (channels with feedback)
- an experimental method to determine which fits best (non linear regression)
Thank you

Comments & Questions

MT = a + b \log(1 + \frac{D}{W})